GENERAL DISCUSSION OF ERRORS

The lab manual contains a good discussion of error estimation and propagation in the introduction on pages 1 - 10. You should ensure you read this section of the lab manual and that you understand how to treat errors. Here I will just emphasize a few points on error propagation and provide a few sample problems (with answers) for you to practice. If you cannot answer the questions in this handout, make sure you ask questions, as the proper treatment of errors is essential to do well on the lab reports. As discussed in the lab manual, suppose we have a function f, which depends on n independent variables, $f = f(x_1, \ldots, x_n)$. In some experiment we will measure all the x_i and determine the error in each measurement, Δx_i. We then want to calculate the error Δf. As explained in the lab manual, the error is

$$\Delta f = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 (\Delta x_i)^2}. \tag{1}$$

If you are interested in more information about this formula see [1] (specifically Ch. 3 around p. 75) or [2] (specifically Ch. 3 around p. 41). Eq. 1 only applies to purely statistical errors. **For PHYS 2260 you should use the more pessimistic formula**

$$\Delta f = \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right| \Delta x_i. \tag{2}$$

Eq. 2 makes the propagation of errors fairly simple, all you need to do is take derivatives. For example, we can now easily derive the error rule for products which you will have used in your first year labs. If $f = cAB$, where c is a constant while A and B are quantities we measure with associated errors ΔA and ΔB, then applying Eq. 2 gives

$$\Delta f = \left|\frac{\partial f}{\partial A}\right| \Delta A + \left|\frac{\partial f}{\partial B}\right| \Delta B$$
$$\Delta f = |c| B \Delta A + |c| A \Delta B$$

so, dividing by f, the percent error, $\varepsilon_f = \Delta f / f$, is

$$\varepsilon_f = \varepsilon_A + \varepsilon_B.$$

You can check that the percent error in the function $f = cA^n B^m$ is $\varepsilon_f = |n|\varepsilon_A + |m|\varepsilon_B$. As practice you should also verify the formulae in the table on p. 9 of the lab manual.

ERROR EXAMPLES

Before the examples, recall that the number of significant figures kept is determined by the error. Once you have calculated the error, keep 1 non zero digit, and then round the actual quantity to the same number of digits. For example suppose we have $A = 1.6784$ m and $\Delta A = 0.00485$ m. Then we would round $\Delta A = 0.005$ m and $A = 1.678$ m so that we would write $A = 1.678 \pm 0.005$ m. Now here are some examples that you should try yourself:

1. Take $f = A + B$ and suppose we have measured $A = 2.0 \pm 0.1$ m and $B = 4.54 \pm 0.05$ m. Then

$$f = A + B = 6.5 \pm 0.2 \text{ m}.$$

2. Now take $f = AB$ and suppose we have $A = 2.0 \pm 0.1$ m $= 2.0$ m$\pm 5\%$ and $B = 4.54 \pm 0.05$ m $=$ 4.54 m $\pm$ 1% then
$$f = AB = 9.08 \text{ m}^2 \pm 6\% = 9.1 \pm 0.5 \text{ m}^2.$$

3. What about raising a number to some power? Let's take $f = A^{\frac{1}{3}}$ with $A = 2.0 \pm 0.1 \text{ m}^3 =$ 2.0 m^3 $\pm$ 5% then
$$f = A^{\frac{1}{3}} = \sqrt[3]{2} \text{ m} \pm \frac{1}{3}5\% = 1.26 \text{ m} \pm 1.7\% = 1.26 \pm 0.02 \text{ m}.$$

4. Now what if we multiply and add at the same time. Lets take $f = 3A^5B + C$ with $A =$ 3.725 ± 0.008 m $= 3.725$ m $\pm$ 0.2 %, $B = 1.45 \pm 0.07$ m $= 1.34$ m $\pm$ 5 % and $C = 254 \pm 3 \text{ m}^6$ $= 254 \text{ m}^6 \pm 1\%$. First we can compute $f = 3(3.725 \text{ m})^5(1.45 \text{ m}) + 254 \text{ m} = 3373.75 \text{ m}^6$. For now I have kept more digits than I will need for my final answer. Remember, you can only do the final rounding at the end, when you know the final error. Now let $X = 3A^5B$. Then we know that
$$\Delta f = \Delta X + \Delta C$$
so we need to find ΔX. From the rule for multiplication and powers
$$\frac{\Delta X}{X} = 5\frac{\Delta A}{A} + \frac{\Delta B}{B}$$
$$\Delta X = X\left(5\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) = 3A^5B\left(5\frac{\Delta A}{A} + \frac{\Delta B}{B}\right)$$
$$\Delta X = 3(3.725 \text{ m})^5(1.45 \text{ m})\left[5(0.002) + 0.05\right] = 187.19 \text{ m}^6$$
Therefore
$$\Delta f = \Delta X + \Delta C = 187.19 \text{ m}^6 + 3 \text{ m}^6 = 190.19 \text{ m}^6 = 200 \text{ m}^6.$$
Notice at the end I rounded the answer so that there was only 1 significant digit. We can now report the full answer including the error
$$f = 3400 \pm 200 \text{ m}^6$$

5. Suppose we measure the speed of light in an experiment
$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}.$$
Assume that the only measured quantity is ϵ_0, then
$$\Delta c = \left|\frac{\partial c}{\partial \epsilon_0}\right|\Delta\epsilon_0$$
$$= \frac{1}{2\sqrt{\mu_0}}\epsilon_0^{-3/2}\Delta\epsilon_0$$
so
$$\Delta c = \left|\frac{\partial c}{\partial \epsilon_0}\right|\Delta\epsilon_0$$
$$= \frac{1}{2\sqrt{\mu_0}}\epsilon_0^{-3/2}\Delta\epsilon_0$$

6. Now suppose you want to calculate the error in the permittivity

$$\epsilon_0 = \frac{Cd}{A + (\kappa - 1)na}$$

(this formula is for the capacitance C of a parallel plate capacitor of area A with plate separation d where the plates are separated by n dielectric spacers of dielectric constant κ and area a, although these details are unimportant) and suppose our measurements are

$$\begin{aligned} C &= 0.08997 \text{ nF } \pm 12.72\% \\ d &= 1.55 \times 10^{-4} \text{ m } \pm 0.645\% \\ A &= 1.201 \pm 0.005 \times 10^{-2} \text{ m}^2 \\ \kappa - 1 &= 2.9 \pm 6\% \\ n &= 3 \\ a &= 3.24 \times 10^{-4} \text{ m}^2 \pm 1.11\% \end{aligned}$$

In the numerator we have Cd and the percent errors for a product add, so the error in the numerator is

$$\Delta(Cd) = 12.72\ \% + 0.645\ \% = 13.365\ \%.$$

Similarly the error in $(\kappa - 1)na$ is

$$\Delta[(\kappa - 1)na] = 6\ \% + 1.11\ \% = 7.11\ \%.$$

Now when we add numbers the errors, not the percent errors add so we add the error in A with the numerical error in $(\kappa - 1)na$ which is

$$\Delta[(\kappa - 1)na] = 0.0711 * 2.9 * 3 * 3.24 \times 10^{-4} = 1.99 \times 10^{-4}.$$

Therefore the total error in the denominator is

$$\Delta[A + (\kappa - 1)na] = \Delta A + \Delta[(\kappa - 1)na] = 0.005 \times 10^{-2} + 0.000199 \times 10^{-2} \sim 0.0052 \times 10^{-2}.$$

Therefore the percent error in the denominator is

$$\frac{\Delta[A + (\kappa - 1)na]}{A + (\kappa - 1)na} = \frac{0.0052}{1.482} = 3.3\ \%.$$

Now when we divide quantities the percent errors add, so the total error in ϵ_0 is

$$\Delta\epsilon_0 = \Delta(Cd) + \Delta[A + (\kappa - 1)na] = 13.365\ \% + 3.3\ \% = 17\ \%$$

Make sure you can follow these examples. Now suppose you want to calculate the error in So what do you need to know? You need to know Eq. 1 (or Eq. 2) and how to use it. Any time you are combining errors this equation can be used and therefore it is really the only rule for combining errors that you need to remember.

QUESTIONS

1. Suppose that to determine the speed of light, c, you had measured both ϵ_0 and μ_0 and had found $\epsilon_0 \pm \Delta\epsilon_0$ and $\mu_0 \pm \Delta\mu_0$. Find the expression for the error in the speed of light.

   Answer:
$$\Delta c = \frac{1}{2\sqrt{\epsilon_0}}\mu_0^{-3/2}\Delta\mu_0 + \frac{1}{2\sqrt{\mu_0}}\epsilon_0^{-3/2}\Delta\epsilon_0$$

2. Use Eq. 2 to find the error in $F = \beta\ln(\alpha\beta) + \rho\cos(\omega) + 5\tau\theta^3\phi^{1/3}$ where $\beta, \alpha, \rho, \omega, \theta, \phi$ and τ are all quantities that we measure with error.

   Answer:
$$\Delta F = (\ln(\alpha\beta) + 1)\,\Delta\beta + \frac{\beta}{\alpha}\Delta\alpha + \cos(\omega)\,\Delta\rho + \rho\sin(\omega)\,\Delta\omega + 5\theta^3\phi^{1/3}\Delta\tau + 15\tau\theta^2\phi^{1/3}\Delta\theta + \frac{5}{3}\tau\theta^3\phi^{-2/3}$$

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[1] J. R. Taylor, *An Introduction To Error Analysis The Study of Uncertainties in Physical Measurements*, 2nd ed. (University Science Books, Sausalito, 1997).

[2] P. R. Bevington and D. K. Robinsein, *Data Reduction and Error Analysis for the Physical Sciences*, 3rd ed. (McGraw-Hill, New York, 2003).